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Spontaneous symmetry breaking on a mutiple-channel hollow cylinder

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ABSTRACT

This Letter investigates coupled asymmetric exclusion processes with two types of particles on multiple parallel channels of a hollow cylinder. The model is inspired by the structure of microtubules, along which motor proteins such as kinesins and dyneins move in opposite directions. Interactions between two-species particles are assumed to take place only on the left and right boundaries where a rule of narrow entrances is applied. Narrow entrances mean that a particle cannot enter the system if either of two nearest-neighbor sites on the neighboring channels is occupied by a particle of the other species. This rule is similar to, but different from, that in [E. Pronina, A.B. Kolomeisky, J. Phys. A 40 (2007) 2275] since the narrow entrance rule in our model involves two neighbors. The phase diagram of our model is studied theoretically and via Monte Carlo simulations. The spontaneous symmetry breaking (SSB) is observed in the system. There are four possible phases in the system and with SSB occurring in two of them: high/low density and asymmetric low/low density. Bulk density and particle currents are also computed. Theoretical calculations deviate from Monte Carlo simulation results due to the neglecting of correlations in particles dynamics in mean-field analysis.

1. Introduction

The investigations of interacting particles out of equilibrium have revealed a rich variety of critical phenomena such as boundary-induced phase transitions, phase separations, and spontaneous symmetry breaking (SSB). Currently, a number of research papers have been devoted to observing SSB in both single- and multiple-channel systems [1–9]. As an exactly solvable paradigmatic model, totally asymmetric simple exclusion process (TASEP) [10] has been used as a modelling tool for the study of SSB in driven diffusive systems.

SSB was firstly observed in a one-dimensional two-species TASEP with open boundary conditions [1]. In this so-called bridge model, it has shown that an asymmetric low-density/low-density (LD/LD) phase and an HD/LD phase can exist in a mean-field approximation, although the symmetric rules are applied to the two species. The analytical results were supported by Monte Carlo simulations. The bridge model has also been investigated with sublattice-parallel update [4]. The SSB phenomenon can probably be explained as a result of an amplification mechanism of fluctuations as indicated in [4]. However, the existence of SSB may still be in controversy as Erickson et al. [2] believe the asymmetric LD/LD phase might be just a finite-size effect. [2] investigated the bridge model via high-precision Monte Carlo data. The simulation results show that the LD/LD phase disappears if the system size is sufficiently large.

In the bridge model, bulk behavior is deterministic but the conditions at the open boundaries are stochastic. Levine and Willmann considered a stochastic bulk by particles attaching to and detaching from the lattice [3]. When two species of particles are assumed to have the same attachment rate and detachment rate, SSB can also be found under some conditions. Popkov et al. [5] extended the bridge model by introducing two junctions to the ordinal bridge model. They found a co-existence region where an HD/LD phase co-exists with a low-density symmetric phase. Note that the dynamics of two-species (e.g., kinesins and dyneins) moving in opposite directions and unbinding (or rebinding) to a one-dimensional lattice (e.g., filament) has also shown SSB [6].

SSB has also been found in a two-channel TASEP system by applying a rule of narrow entrances [7]. In this rule, particles cannot enter a channel (e.g., channel 1) if the exit site in the other channel (e.g., channel 2) is occupied. Particles on the two channels can be seen as two species of particles: one species go to one direction along channel 1; the other go to the opposite direction along channel 2. One can see that interactions between two-species particles can only take place on the left and right boundaries. It was found that the HD/LD phase and asymmetric LD/LD phase can exist in the system, which reveal the existence of SSB. The above-mentioned investigations of SSB are based on the random update.

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SSB is also observed in a two-species two-channel system in parallel update [8]. When the narrow entrance rule [7] is applied, two symmetry-breaking phases (HD/LD and LD/LD) can be found. The investigations [7–9] show that different update procedures (e.g., random and parallel) for the same system can all result in SSB.

In this Letter, a novel multiple-channel TASEP model on a hollow cylinder with narrow entrances is presented. The model is inspired by the structure of microtubules, along which motor proteins such as kinesins and dyneins move in opposite directions. Interactions between two-species particles are assumed to take place only on the left and right boundaries where a rule of narrow entrances is applied. A microtubule can be seen as a hollow cylinder, which consists of many filaments, while each filament can be viewed as a channel. In our model, each channel has two nearest neighbors. The states of the exit sites of both left and right neighbor channels are considered when a particle (kinesins or dyneins) enters into a filament with narrow entrances. Note that the channel has only one neighbor in the original narrow entrance models in [7–9]. It is necessary to investigate the traffic dynamics of TASEPs with two-neighbor interactions at the boundaries and to check if SSB exists in such a system. In this Letter, we study the random update only, although extending this work to other updating procedures is possible.

The Letter is organized as follows. Section 2 gives a brief description of our model and theoretical analysis. In Section 3, the results of Monte Carlo simulations are discussed. Our conclusions are given in Section 4.

2. Model and theoretical analysis

The system consists of M parallel one-dimensional channels with two types of species (e.g., positive and negative) particles moving in different channels in opposite directions and each channel has L sites (see Fig. 1(a)). Hopping between channels is not allowed. In the bulk, particles can hop to the next sites with rate 1 provided the next sites are empty. At the exit sites, particles are removed with rate β. At the entrance sites, particles are injected with rate α provided these sites and the nearest sites on both left and right neighbor channels are empty. We label a channel as C_i (the i-th channel) where $i = 1, 2, \ldots, M$. C_i has two neighbor channels:

- When $i = 1$, the neighbors of C_1 are C_2 and C_M.
- When $1 < i < M$, the neighbors of C_i are C_{i-1} and C_{i+1}.
- When $i = M$, the neighbors of C_M are C_1 and C_{M-1}.

The positive particles are assumed to move from the left end to the right end of the cylinder on the *odd channels* (e.g., $i = 1, 3, 5, \ldots$), while the negative particles are assumed to move from the right to the left on the *even channels* (e.g., $i = 2, 4, 6, \ldots$). Also, see Fig. 2. Obviously, there are two different types of systems. When M is an *even number*, particles move on two adjacent channels in opposite directions (see Fig. 1(a)). In this case, the channels of the cylinder are *symmetric*. When M is an odd number, C_1 and C_M can be seen as the boundaries and particles move on C_1 and C_M in the same direction. Thus, the cylinder is not symmetric and it can be tiled into a rectangle with M channels (see Fig. 1(b)). Therefore, in this Letter, we only consider the symmetric case with even number of channels.

We firstly recall the results of a usual TASEP on a single channel with open boundary conditions with random update [11,12]. When entrance rate $\alpha \geqslant 1/2$ and exit rate $\beta \geqslant 1/2$, the system is in a maximal-current (MC) phase with particle current $J_{MC} = 1/4$ and bulk density $\rho_{bulk,MC} = 1/2$; When $\alpha > \beta$ and $\beta < 1/2$, the system is in a high density (HD) phase with $J_{HD} = \beta(1-\beta)$ and $\rho_{bulk,HD} = 1 - \beta$; When $\alpha < \beta$ and $\alpha < 1/2$, the system is in a low density (LD) phase with $J_{LD} = \alpha(1-\alpha)$ and $\rho_{bulk,LD} = \alpha$.

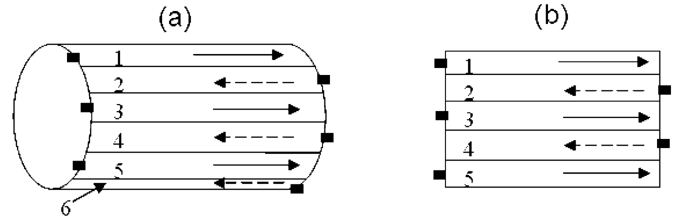

Fig. 1. Sketch of TASEPs on a cylinder with narrow entrances. (a) The number of channels is even; (b) The number of channels is odd, e.g., $M = 5$. The filled rectangles correspond to narrow entrances.

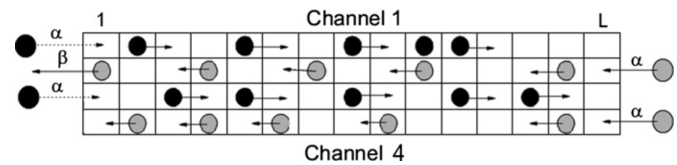

Fig. 2. Illustration of multiple-channel TASEPs on a cylinder with narrow entrances. Positive particles move from the left to the right, while negative particles move from the right to the right. Black-line arrows at boundaries represent allowed entrances with rate α. Dash–dot-line arrows mean prohibited entrances.

We assume that the effective entrance rates are given by α_1 and α_2, respectively, for the all *odd* and all *even* number channels. n_1 and m_L are average densities at the first sites and the last sites of the *even* and *odd* number channels, respectively. Therefore, assuming that there are no correlations in occupancies of neighboring channels,

$$\alpha_1 = \alpha(1-n_1)^2, \qquad \alpha_2 = \alpha(1-m_L)^2. \tag{1}$$

The physical meaning for these expressions is that a particle can enter a channel only when the corresponding sites at the neighboring channels are empty. We then define the exit current from the *odd* and *even* number channels, respectively, as:

$$J_1 = \beta m_L, \qquad J_2 = \beta n_1. \tag{2}$$

When the system is in the symmetric phase, there are three possible stationary phases: LD, HD, and MC. Also, all properties in these channels are identical. They are:

$$J_1 = J_2, \qquad \alpha_1 = \alpha_2. \tag{3}$$

In the MC phase, the current is equal to $J_1 = J_2 = 1/4$. According to Eq. (2), one obtains $n_1 = 1/(4\beta)$. Then, the effective entrance rate can be rewritten as

$$\alpha_1 = \alpha\left(1 - \frac{1}{4\beta}\right)^2. \tag{4}$$

When the MC phase exists, the boundary conditions are: $\alpha_1 \geqslant 1/2$ and $\beta \geqslant 1/2$. Correspondingly, α should satisfy

$$\alpha \geqslant \frac{8\beta^2}{16\beta^2 - 8\beta + 1}. \tag{5}$$

In the HD phase, the following conditions should be satisfied

$$\alpha_1 > \beta, \qquad \beta < \frac{1}{2}. \tag{6}$$

The stationary current in this phase is determined by exit rate β. That is, $J_1 = J_2 = \beta(1-\beta)$. Compared with Eq. (2), we have $n_1 = 1 - \beta$. Eq. (1) can be rewritten as $\alpha_1 = \alpha\beta^2$. According to Eq. (6), we obtain $\alpha\beta > 1$, and this is impossible for $\alpha \in [0, 1]$ and $\beta \in$

[0, 1]. We thus confirm that the HD phase does not exist in the symmetric system.

In the LD phase, the current is $J_1 = \alpha_1(1-\alpha_1)$. According to Eqs. (1)–(3), we have the following equation:

$$\alpha\alpha_1^4 - 2\alpha\alpha_1^3 + (2\alpha\beta + \alpha)\alpha_1^2 - (2\alpha\beta + \beta^2)\alpha_1 + \alpha\beta^2 = 0. \tag{7}$$

The exact solution for the above equation can be obtained. However, the results are extremely bulky (more than one page). The result can be found in [13].

Note that if $\alpha_1 = 1/2$ in Eq. (7), one can obtain $\alpha = 8\beta^2/(16\beta^2 - 8\beta + 1)$. This is exactly the same as that we can obtained from Eq. (5). Also, in Eq. (5), if $\alpha_1 > 1/2$, $\alpha > 8\beta^2/(16\beta^2 - 8\beta + 1)$. Therefore, one can get a hypothesis that the condition for $\alpha_1 < 1/2$ for Eq. (7) could be:

$$\alpha < \frac{8\beta^2}{16\beta^2 - 8\beta + 1}. \tag{8}$$

The hypothesis has been tested numerically. We found that the above equation is only the necessary condition for $\alpha_1 < 1/2$. It is not the sufficient condition. This is different from the relationship in Ref. [7] (refer to Eqs. (10) and (13) of the reference).

For the asymmetric phases, the currents and density profiles in odd and even number channels are not equal, namely

$$J_1 \neq J_2, \qquad \alpha_1 \neq \alpha_2. \tag{9}$$

There are six possible phases: asymmetric LD/LD, asymmetric HD/HD, asymmetric MC/MC, LD/HD, LD/MC, and HD/MC phases. The LD/HD phase means that the odd (even) number channels are in the LD phase, while the even (odd) number channels are in the HD phase. For the asymmetric LD/LD phase, it can exist if

$$\alpha_1 < \beta, \qquad \alpha_2 < \beta, \qquad \alpha_1 < \frac{1}{2}, \qquad \alpha_2 < \frac{1}{2}. \tag{10}$$

Since both channels are in LD, we have

$$J_1 = \alpha_1(1-\alpha_1), \qquad J_2 = \alpha_2(1-\alpha_2). \tag{11}$$

J_1 and J_2 can also be calculated by

$$J_1 = m_L\beta, \qquad J_2 = n_1\beta. \tag{12}$$

Therefore, we have

$$m_L = \frac{\alpha_1(1-\alpha_1)}{\beta}, \qquad n_1 = \frac{\alpha_2(1-\alpha_2)}{\beta}. \tag{13}$$

Therefore, α_1 and α_2 are determined by

$$\alpha_1 = \alpha(1-n_1)^2 = \alpha\left[1 - \frac{\alpha_2(1-\alpha_2)}{\beta}\right]^2 < \beta, \tag{14}$$

$$\alpha_2 = \alpha(1-m_L)^2 = \alpha\left[1 - \frac{\alpha_1(1-\alpha_1)}{\beta}\right]^2 < \beta. \tag{15}$$

The exact solutions (involving only the arithmetical operations and radicals) for the above equations are impossible to obtain when the equations are transformed into an univariate polynomials with power $\geqslant 5$ according to Abel–Ruffini theorem (also known as Abel's impossibility theorem). Thus, we have to apply Newton's method to Eqs. (7)–(8), which allows us to calculate approximately the asymmetric LD/LD phase region.

In the LD/HD phase, the corresponding conditions are

$$\alpha_1 < \beta, \qquad \alpha_2 > \beta, \qquad \alpha_1 < \frac{1}{2}, \qquad \beta < \frac{1}{2}. \tag{16}$$

The currents in this phase are given by

$$J_1 = \alpha_1(1-\alpha_1) = m_L\beta \quad \text{and} \quad J_2 = \beta(1-\beta) = \beta n_1. \tag{17}$$

α_1 and α_2 are solved as

$$\alpha_1 = \alpha\beta^2, \qquad \alpha_2 = \alpha\left[1 - \alpha\beta\left(1 - \alpha\beta^2\right)\right]^2. \tag{18}$$

The conditions $\alpha_1 < \beta$, $\alpha_1 < 1/2$, and $\beta < 1/2$ in Eq. (14) can be modified to $\alpha\beta < 1$ and $\alpha\beta^2 < 1/2$, respectively, which could be satisfied for $\alpha \in [0, 1]$ and $\beta \in [0, 1]$. For the conditions $\alpha_2 = \alpha[1 - \alpha\beta(1 - \alpha\beta^2)]^2 > \beta$ and $\beta < 1/2$, again, we have to apply Newton's method to this equation. Our numerical result confirms the existence of this phase.

The LD/MC phase is specified by

$$\alpha_1 < \beta, \qquad \alpha_1 < \frac{1}{2}, \qquad \alpha_2 > \frac{1}{2}, \qquad \beta > \frac{1}{2}. \tag{19}$$

The corresponding currents in the system can be written as

$$J_1 = \alpha_1(1-\alpha_1) = m_L\beta, \quad J_2 = \beta n_1 = \frac{1}{4}, \tag{20}$$

which leads to the following expressions

$$\alpha_1 = \alpha\left(1 - \frac{1}{4\beta}\right)^2,$$
$$\alpha_2 = \alpha\left\{1 - \frac{\alpha}{\beta}\left(1 - \frac{1}{4\beta}\right)^2\left[1 - \alpha\left(1 - \frac{1}{4\beta}\right)^2\right]\right\}^2. \tag{21}$$

Again, the exact solution cannot not been solved. Newton's method has to be applied to examine if the LD/MC phase exists or not. The numerical result shows the LD/MC phase does not exist.

The conditions for the existence of the HD/MC phase are

$$\alpha_1 > \beta, \qquad \beta < \frac{1}{2}, \qquad \alpha_2 > \frac{1}{2}, \qquad \beta > \frac{1}{2}. \tag{22}$$

It is found that these conditions conflict each other. This suggests that the HD/MC phase does not exist.

In the asymmetric HD/HD phase, the currents are determined by β, that is,

$$J_1 = \beta(1-\beta) = m_L\beta, \qquad J_2 = \beta(1-\beta) = \beta n_1, \tag{23}$$

which leads to $n_1 = m_L = 1 - \beta$, and $\alpha_1 = \alpha_2 = \alpha\beta^2$. This is inconsistent with Eq. (7). Thus, the asymmetric HD/HD phase does not exist in the system.

Clearly, the asymmetric MC/MC phase does not exist in the system since the current is constant and independent of α and β in the MC phase. Thus, there is no asymmetric MC/MC. Therefore, the resulting possible phases are: the asymmetric LD/LD, LD/HD, symmetric LD, and symmetric MC phases. Fig. 3 shows the phase diagrams obtained via theoretical analysis and simulations. The theoretical mean-field analysis results for asymmetric phases are obtained mainly through Newton's method.

It can be seen that the phase structure from Monte Carlo simulation slightly deviates from the mean-field results, in particular, for large values of α. This implies that correlations between particles in the left and right boundaries are important in the dynamics of such a system. Similar observations have been found in [7,8].

There are several differences which can be observed from phase diagrams in our model and that in Fig. 2 in [7]. Firstly, the asymmetric LD/HD region in our model is larger than that in [7] (see Fig. 3(b)). Correspondingly, the symmetric LD region in our model is smaller then that in [7]. Secondly, the MC region in our model is smaller than that in [7]. Finally, the region for existence of the asymmetric LD/LD phase in our model is larger than that in [7]. This could be due to the fact that two neighboring channels have stronger overall effect on the dynamics in the given channel.

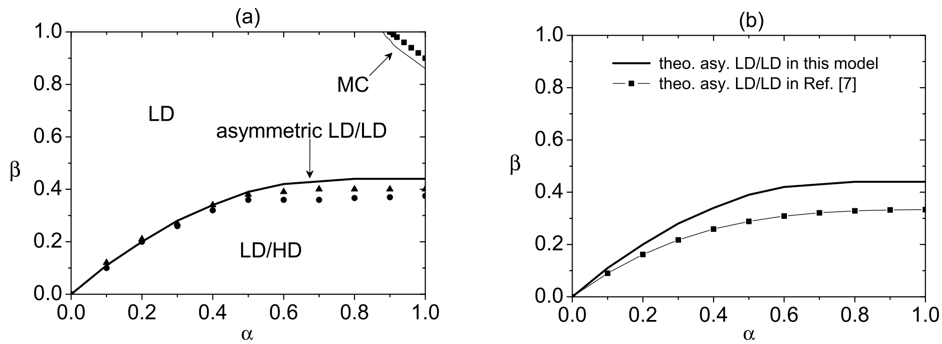

Fig. 3. (a) Phase diagram of the system for $M = 8$ and $L = 1000$. Solid lines are the theoretical results, while symbols correspond to simulation results. Filled triangles represent the boundary between the LD phase and the asymmetric LD/LD phase, while filled circles correspond to the boundary between the LD/HD phase and the asymmetric LD/LD phase. (b) Comparison of theoretical predictions in the asymmetric LD/LD phase between our model and the model in [7] where each channel has only one neighboring channel.

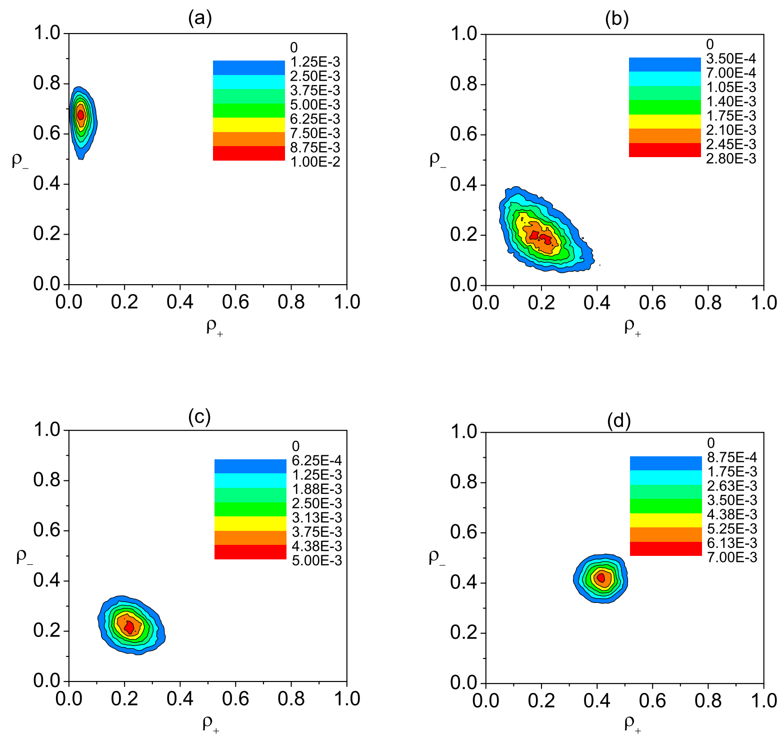

Fig. 4. Histograms of particle densities at different phases. (a) The HD/LD phase with $\alpha = 0.6$ and $\beta = 0.3$; (b) the asymmetric LD/LD phase with $\alpha = 0.6$ and $\beta = 0.4$; (c) the symmetric LD phase with $\alpha = 0.6$ and $\beta = 0.5$; and (d) the MC phase with $\alpha = 1.0$ and $\beta = 0.95$.

3. Simulations and discussions

The histograms $P_L(\rho_+, \rho_-)$ of particle densities are firstly simulated, where ρ_+ and ρ_- are instantaneous densities of particles in odd and even number channels, respectively. The system size $L = 1000$ and the number of channels $M = 8$ are used unless otherwise mentioned. In simulations, stationary density profiles and currents are obtained by averaging 8×10^8 time steps. The first 10^8 time steps are discarded as transients.

Three typical particle density histograms in the HD/LD phase, asymmetric LD/LD phase, and symmetric LD phase are shown in Fig. 4. It can be seen that a double peak with two off-diagonal maxima appears in the HD/LD phase, while a single peak exists on the diagonal in the symmetric LD and asymmetric LD/LD phases. The transition between the two asymmetric phases is marked by histograms with two long ridges, one running close to the ρ_+-axis, and the other close to ρ_--axis, which has been indicated in [2].

The flipping processes are then simulated in order to investigate the asymmetric phases obtained from the mean-field approximation. The density difference $\rho_+ - \rho_-$ has been measured as functions of time (see Fig. 5). The system flips between positive net values and negative net values. The positive (negative) net values imply that the bulk density and current of positive (negative) particles in odd (even) number channels are larger than that of negative (positive) particles in even (odd) number channels. The flipping processes are observed clearly. It is found that the flipping processes are qualitatively identical for the same phase but with different number of channels (see Figs. 5(a)–(b) and 5(c)–(d)). This suggests the existence of SSB in the system.

We also compare average currents obtained from simulation results in our model and in [7] (see Fig. 6). The average current is represented as the system current divided by the number of channels. For simplicity, we assume arbitrary that the value of α is fixed, while the value of β changes from 0 to 1. For a fixed α, the current in our model is less than that in [7]. This is the result that particles enter our system more difficult than that in [7].

In order to study the finite-size effect in our model we performed computer simulations with different system length (up to

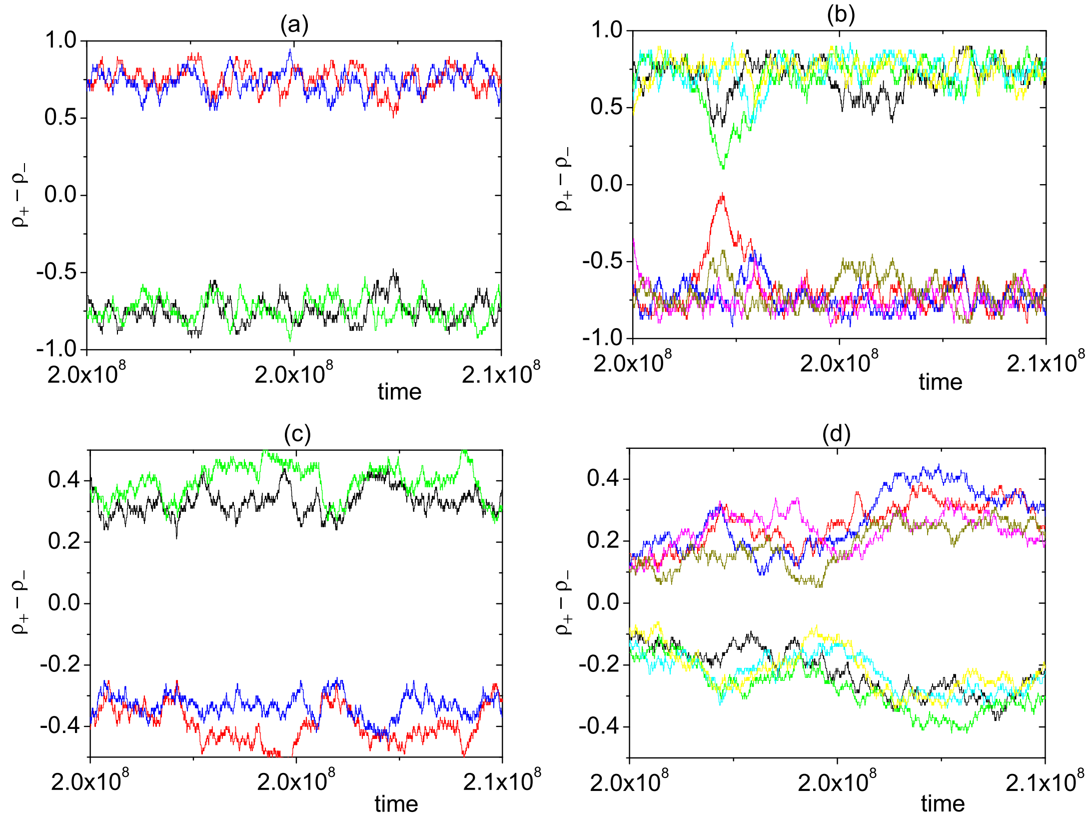

Fig. 5. (Color online.) (a) and (b) The flipping process in the HD/LD phase with parameters: $N = 40$, $\alpha = 0.5$, and $\beta = 0.2$; (c) and (d) The flipping process in the asymmetric LD/LD phase with parameters: $N = 100$, $\alpha = 0.9$, and $\beta = 0.4$. (a) and (c) $M = 4$; (b) and (d) $M = 8$.

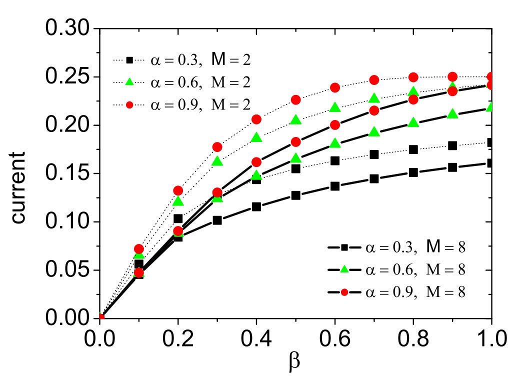

Fig. 6. (Color online.) Currents obtained from simulations in our model (thick solid lines with symbols) and in [7] (dotted lines with symbols) at $\alpha = 0.3$, 0.6, and 0.9.

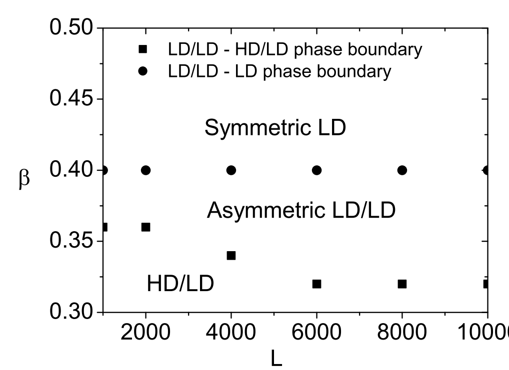

Fig. 7. The finite-size effect in our model: the phase boundary between the asymmetric LD/LD and symmetric LD phases seems independent of the system size.

$L = 10\,000$). Phase boundaries have been confirmed and shown in Fig. 7. It is found that the phase boundary between the asymmetric LD/LD and symmetric LD phases does not depend on the system size, while the region of the asymmetric LD/LD phase seems to expand and keeps unchanged with the further increase of the system size. This suggests that the asymmetric LD/LD phase probably exists in the thermodynamic limit ($L \to \infty$).

Note that the status of the LD/LD symmetry broken phases is still controversial for the bridge model. Erickson et al. [2] suggested that for some parameters the phase would disappear. However, Clincy et al. [14] indicated that the LD/LD symmetry broken phase does exist in the thermodynamic limit. Our results appear to support this. Also, note that Fig. 7 in this Letter seems qualitatively different from Fig. 3 in [7]. This may be caused by the fact that all entrances have two neighbor entrances in the current model. Obviously, further investigations of this phase are required in order to better understand symmetry breaking phenomena.

4. Conclusions

The totally asymmetric simple exclusion processes (TASEPs) on a hollow cylinder with narrow entrances is investigated under random update. The model is motivated by the structure of microtubules as well as the dynamics of protein motors (i.e., kinesins and dyneins) moving along microtubules in opposite directions. The system is investigated by using a mean-field theory and verified by Monte Carlo simulations. There are two possible symmetry breaking phases, i.e., HD/LD and asymmetric LD/LD, in the system. The phase structure obtained from Monte Carlo simulations deviates from the mean-field results. This is probably due to neglecting correlations in particle dynamics of this system.

In order to illustrate the density profiles of the system, particle density histograms $P_L(\rho_+, \rho_-)$ are investigated. The flipping processes are observed and exhibit qualitatively identical for the same phase but with different number of channels. The finite-size effects in this model are studied, which also suggests the existence of the spontaneous symmetry breaking in the proposed model. We also compare the results with that obtained from two-channel TASEPs with narrow entrances in random update [7]. The current is lower due to the narrower entrances in our model.

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