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# Exact solutions for a partially asymmetric exclusion model with two species

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**Abstract**

We present a partially asymmetric exclusion model with two species. It consists of a one-dimensional lattice with two types of particles hopping with different rates in both directions. Exact solutions are given for some values of the parameters. A phase diagram is constructed. The model is found to show two phases, one of them a phase with a maximal current of the particles.

*Keywords*: Asymmetric simple exclusion process; Steady state; Phase transition; Spontaneous symmetry breaking
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## 1. Introduction

Recently, much attention has been focused on asymmetric simple exclusion processes (ASEPs) as simple models in non-equilibrium statistical mechanics. A one-dimensional ASEP is a model of particles hopping stochastically and interacting through hard-core exclusion. It is connected to a series of physical phenomena such as gel electrophoresis [1], interface growth [2], and the dynamics of shocks [3]. Although the models are quite simple to formulate there are only a few exact results available [4,5].

The most successful method in solving ASEP models exactly is the so-called matrix product ansatz [4–7]. In this approach probabilities of different states of the system are represented by a product of non-commuting matrices. Within this approach the particle currents, densities of particles and correlation functions can be obtained exactly.

Evans, Foster, Godreche and Mukamel (EFGM) introduced a totally asymmetric one-dimensional exclusion model with two species [8]. In this model particles can hop only in one direction ("+" particles to the right, "−" particles to the left). Using the matrix product method they found exact solutions for some sets of parameters. They also calculated the phase diagram of the model in the mean-field approximation and


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by Monte Carlo simulations. EFGM reported a phenomenon of spontaneous symmetry breaking corresponding to the existence of phases where the currents of the two species of particles are not equal.

The present paper introduces and treats a partially asymmetric one-dimensional exclusion model with two species of particles hopping along the open chain. In this model particles can move in both directions but with different hopping rates. For convenience we call the particles coming from the left end "left" or "+" particles, and particles coming from the right end "right" or "−" particles. Each site of a one-dimensional lattice of length $N$ may be occupied by a left particle or by a right particle or be empty. Stochastic dynamical rules govern the evolution of the system. During the infinitesimal time step $dt$, any nearest-neighbor pair of sites $i, i+1 (1 \leqslant i \leqslant N-1)$ evolves as follows:

$$(+)_i(0)_{i+1} \to (0)_i(+)_{i+1} \quad \text{with rate } p,$$

$$(0)_i(+)_{i+1} \to (+)_i(0)_{i+1} \quad \text{with rate } s,$$

$$(0)_i(-)_{i+1} \to (-)_i(0)_{i+1} \quad \text{with rate } p,$$

$$(-)_i(0)_{i+1} \to (0)_i(-)_{i+1} \quad \text{with rate } s,$$

$$(+)_i(-)_{i+1} \to (-)_i(+)_{i+1} \quad \text{with rate } q.$$

Here we always assume that the positive (negative) particles move preferably to the right (left), i.e. $p \geqslant s$.

The left particles (or "+") are injected at the left end and removed from the right end. The right particles (or " − ") enter the system at the right end and leave it from the left end. Thus in each infinitesimal time step $dt$ the following events may occur at the boundaries :
At the left end ($i = 1$)

$$(0)_1 \to (+)_1 \text{ with rate } \alpha,$$

$$(-)_1 \to (0)_1 \text{ with rate } \beta.$$

At the right end ($i = N$)

$$(0)_N \to (-)_N \text{ with rate } \alpha,$$

$$(+)_N \to (0)_N \text{ with rate } \beta.$$

For the special case $p = 1$ and $s = 0$ the model reduces to the totally asymmetric exclusion model with two species studied by EFGM [8].

We use the matrix product method to solve this model exactly for certain sets of parameters. It enables us to construct a phase diagram and to compare it with that for the earlier EFGM model in which $p = 1$ and $s = 0$.

## 2. Matrix product method

Following EFGM [8] we introduce two occupation numbers, $l_i$ and $r_i$, for each site $i$, where $l_i = 1$ if site $i$ is occupied by a left ("+") particle and 0 otherwise. Similarly, $r_i = 1$ if site $i$ is occupied by a right ("−") particle and 0 otherwise. Any configuration of the system is uniquely determined by the set of occupation numbers $\{l_i, r_i\}$. We are looking for a long-time limit solutions when the system reaches a steady state. It implies that all the probabilities $P_N(\{l_1,r_1\},\ldots,\{l_N,r_N\})$ of finding the system in configurations $(\{l_1,r_1\},\ldots,\{l_N,r_N\})$ are stationary, i.e. satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}P_N(\{l_1,r_1\},\ldots,\{l_N,r_N\}) = 0. \tag{1}$$

It is more convenient to use unnormalized weights $f_N(\{l_i,r_i\})$ defined as

$$P_N(\{l_i,r_i\}) = \frac{f_N(\{l_i,r_i\})}{Z_N}, \tag{2}$$

where

$$Z_N = \sum_{\{l_i,r_i\}} f_N(\{l_i,r_i\}). \tag{3}$$

Then the matrix product method suggests that $f_N(\{l_i,r_i\})$ may be constructed as

$$f_N(\{l_i,r_i\}) = \langle W \mid \prod_{i=1}^{N}[l_i\mathbf{L} + r_i\mathbf{R} + (1 - l_i - r_i)\mathbf{E}] \mid V\rangle \tag{4}$$

Eq. (4) means that the weight $f_N(\{l_i,r_i\})$ is given by a product of $N$ matrices (**L**, **R**, or **E**) with matrix **L** at position $i$ if site $i$ is occupied by a left particle ($l_i = 1$), a matrix **R** at position $i$ if this site is occupied by a right particle ($r_i = 1$) and a matrix **E** if this site is empty ($l_i = 0, r_i = 0$). The multiplication by the vectors $\mid V\rangle$ and $\langle W \mid$ on the matrix product produces a scalar value for $f_N$.

The dynamics of the system is governed by a Master equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}P_N(\tau_1,\tau_2,\ldots,\tau_N) = &\sum_{\sigma_1}(h_1)_{\tau_1;\sigma_1}P_N(\sigma_1,\tau_2,\ldots,\tau_N)\\ &+\sum_{i=1}^{N-1}\sum_{\sigma_i,\sigma_{i+1}}(h)_{\tau_i,\tau_{i+1};\sigma_i,\sigma_{i+1}}P_N(\tau_1,..,\sigma_i,\sigma_{i+1},\ldots,\tau_N)\\ &+\sum_{\sigma_N}(h_N)_{\tau_N;\sigma_N}P_N(\tau_1,..,\tau_{N-1},\sigma_N)\,.\end{aligned} \tag{5}$$

Here $\tau_i$ (or $\sigma_i$) indexes the state of site $i$, with $\tau_i$ (or $\sigma_i$)$= +1, -1$, or 0 when there is a left particle at site $i$, when there is a right particle at site $i$, or when site $i$ is empty, respectively. The matrices $h_1, h$, and $h_N$ represent the transition rates, i.e. $(h)_{\tau_i,\tau_{i+1};\sigma_i,\sigma_{i+1}}$ is the non-diagonal element of the matrix corresponding to the transition rate from the configuration $(\tau_1,..,\sigma_i,\sigma i + 1,\ldots,\tau_N)$ to the configuration $(\tau_1,..,\tau_i,\tau_{i+1},\ldots,\tau_N)$, and $-(h)_{\tau_i,\tau_{i+1};\tau_i,\tau_{i+1}}$ is the transition rate out of the configuration $(\tau_1,..,\tau_i,\tau_{i+1},\dots,\tau_N)$. Using the stochastic rules we can construct these matrices. Then, the only non-zero elements of the transition rate matrices are

$$
\begin{aligned}
&(h_1)_{0;-1} = -(h_1)_{-1;-1} = \beta,\\
&(h_1)_{+1;0} = -(h_1)_{0;0} = \alpha,\\
&(h)_{-1,0;0,-1} = -(h)_{0,-1;0,-1} = p,\\
&(h)_{0,+1;+1,0} = -(h)_{+1,0;+1,0} = p,\\
&(h)_{0,-1;-1,0} = -(h)_{-1,0;-1,0} = s,\\
&(h)_{+1,0;0,+1} = -(h)_{0,+1;0,+1} = s,\\
&(h)_{-1,+1;+1,-1} = -(h)_{+1,-1;+1,-1} = q,\\
&(h_N)_{-1;0} = -(h_1)_{0;0} = \alpha,\\
&(h_N)_{0;+1} = -(h_1)_{+1;+1} = \beta.
\end{aligned} \tag{6}
$$

Let us assume that there exist three coefficients $x_{-1}, x_0$, and $x_{+1}$ such that the following conditions are satisfied for each choice of $\tau_i$:

$$\sum_{\sigma_1}(h_1)_{\tau_1;\sigma_1}P_N(\sigma_1,\tau_2,\dots,\tau_N) = x_{\tau_1}P_{N-1}(\tau_2,\dots,\tau_N), \tag{7}$$

$$
\begin{aligned}
&\sum_{\sigma_i,\sigma_{i+1}}(h)_{\tau_i,\tau_{i+1};\sigma_i,\sigma_{i+1}}P_N(\tau_1,..,\sigma_i,\sigma_{i+1},\dots,\tau_N)\\
&\quad = -x_{\tau_i}P_{N-1}(\tau_1,..,\tau_{i-1},\tau_{i+1},..,\tau_N) + x_{\tau_{i+1}}P_{N-1}(\tau_1,..,\tau_i,\tau_{i+2},..,\tau_N),
\end{aligned} \tag{8}
$$

$$\sum_{\sigma_N}(h_N)_{\tau_N;\sigma_N}P_N(\tau_1,..,\tau_{N-1},\sigma_N) = -x_{\tau_N}P_{N-1}(\tau_1,\dots,\tau_{N-1}). \tag{9}$$

If such coefficients $x_{-1}, x_0$, and $x_{+1}$ exist, then the probabilities $P_N$ (or $f_N$) are steady state quantities ($\mathrm{d}P_N/\mathrm{d}t = 0$), as can be checked by adding Eqs. (7)–(9) and comparing the result with Eq. (5). When we replace $P_N$ (or $f_N$) by their Eqs. (2)–(4) and substitute into Eqs. (7)–(9), we obtain the following conditions on matrices **L**, **R** and **E**:

$$
\begin{aligned}
&x_{-1} + x_0 + x_{+1} = 0,\\
&\beta\langle W \mid \mathbf{R} = x_{-1}\langle W \mid,\\
&\alpha\langle W \mid \mathbf{E} = x_{+1}\langle W \mid,\\
&\alpha\mathbf{E} \mid V\rangle = -x_{-1} \mid V\rangle,\\
&-\beta\mathbf{L} \mid V\rangle = -x_{+1} \mid V\rangle,\\
&p\mathbf{ER} - s\mathbf{RE} = -x_{-1}\mathbf{E} + x_0\mathbf{R},\\
&p\mathbf{LE} - s\mathbf{EL} = x_{+1}\mathbf{E} - x_0\mathbf{L},\\
&q\mathbf{LR} = -x_{-1}\mathbf{R} + x_{+1}\mathbf{L}.
\end{aligned} \tag{10}
$$

The choice of the coefficients $x_{-1}, x_0, x_{+1}$ is almost free (subject only to the first of Eq. (10)) [4]; we take $x_0 = 0$, $x_{-1} = -q$, and $x_{+1} = q$. Then the matrix algebra

(Eq. (10)) simplifies to the following equations:

$$\langle W \mid \mathbf{R} = \frac{q}{\beta}\langle W \mid, \tag{11}$$

$$\langle W \mid \mathbf{E} = \frac{q}{\alpha}\langle W \mid, \tag{12}$$

$$p\mathbf{ER} - s\mathbf{RE} = q\mathbf{E}, \tag{13}$$

$$p\mathbf{LE} - s\mathbf{EL} = q\mathbf{E}, \tag{14}$$

$$\mathbf{LR} = \mathbf{L} + \mathbf{R}, \tag{15}$$

$$\mathbf{E} \mid V\rangle = \frac{q}{\alpha} \mid V\rangle, \tag{16}$$

$$\mathbf{L} \mid V\rangle = \frac{q}{\beta} \mid V\rangle. \tag{17}$$

Rescaling of the hopping rates and time with $p$ and redefinitions of $q/p$, $s/p$, $\alpha/p$, $\beta/p$ in the above equations as $q$, $s$, $\alpha$, $\beta$, respectively, give Eqs. (11)–(17) with $p = 1$. Thus, without loss of generality, we assume $p = 1$ in the following discussions.

If we can find such matrices **L**, **R** and **E** and vectors $\mid V\rangle$ and $\langle W \mid$, then the densities of positive (left) and negative (right) particles can be calculated exactly [4,8] as follows:

$$\langle l_i\rangle = \frac{\langle W \mid \mathbf{G}^{i-1}\mathbf{L}\mathbf{G}^{N-i} \mid V\rangle}{\langle W \mid \mathbf{G}^{N} \mid V\rangle}, \tag{18}$$

$$\langle r_i\rangle = \frac{\langle W \mid \mathbf{G}^{i-1}\mathbf{R}\mathbf{G}^{N-i} \mid V\rangle}{\langle W \mid \mathbf{G}^{N} \mid V\rangle}, \tag{19}$$

where

$$\mathbf{G} = \mathbf{L} + \mathbf{R} + \mathbf{E}. \tag{20}$$

Also the currents of positive and negative particles can be calculated explicitly. Taking the positive direction of the current for the left particles to be from left to right, and for the right particles to be from right to left, the matrix Eqs. (12) and (16) imply for the currents:

$$\begin{aligned} J^{+} &= \alpha\frac{\langle W \mid \mathbf{E}\mathbf{G}^{N-1} \mid V\rangle}{\langle W \mid \mathbf{G}^{N} \mid V\rangle} = q\frac{\langle W \mid \mathbf{G}^{N-1} \mid V\rangle}{\langle W \mid \mathbf{G}^{N} \mid V\rangle}, \\ J^{-} &= \beta\frac{\langle W \mid \mathbf{R}\mathbf{G}^{N-1} \mid V\rangle}{\langle W \mid \mathbf{G}^{N} \mid V\rangle} = q\frac{\langle W \mid \mathbf{G}^{N-1} \mid V\rangle}{\langle W \mid \mathbf{G}^{N} \mid V\rangle}. \end{aligned} \tag{21}$$

Thus, formally the solution of the problem reduces to finding a set of matrices and vectors satisfying the matrix Eqs. (11)–(17). By using the matrix product ansatz the exact solutions are found in the following cases:

### 2.1. Solution for $\alpha = \infty$

The solution in this case is the same as that for the totally asymmetric exclusion model with two species treated by EFGM [8]. The matrix $\mathbf{E}$ can then be thought of as consisting only of zeros. It implies that the probability to find a hole at any site in any configuration is zero. The problem effectively reduces to an ASEP with only one type of particle. For this case all the results can be found from [4].

In the limit $N \to \infty$ there are two phases. In one, $q \leqslant 2\beta$ and the current is given by

$$J^+ = J^- = q/4 . \tag{22}$$

This is the maximal-current phase, i.e. the current in this phase is greater than that in the second phase, and depends on only one parameter $q$.

In the second, $q > 2\beta$ and the current is given by [1]

$$J^+ = J^- = \beta(1 - \beta/q). \tag{23}$$

### 2.2. Solution for $s = 1 - \beta$ and $q = 2\beta$

For this set of parameters and for all others we will choose the vectors $\langle W \mid$ and $\mid V\rangle$ as follows:

$$\langle W \mid = (1,0,0,\ldots), \mid V\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}. \tag{24}$$

There is a simple one-dimensional matrix representation which satisfies Eqs. (11)–(17) for the given set of parameters. We may take $\mathbf{E} = 2\beta/\alpha$, and $\mathbf{R} = \mathbf{L} = 2$. Then the densities of left and right particles and the currents are independent both of the site position $i$ and the lattice size $N$

$$\langle l_i\rangle = \langle r_i\rangle = \frac{\alpha}{2\alpha + \beta},$$

$$J^+ = J^- = \frac{\alpha\beta}{2\alpha + \beta}. \tag{25}$$

We have here only one phase with the current given by Eq. (25). It is not the maximal-current phase. When $s = 0$ ($\beta = 1$) we recover the results obtained by EFGM [8].

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[1] There is apparently a typographical error in the formula for the current in this phase reported by EFGM [8, p. 74].

### 2.3. Solution for $s = 1 - \beta$ and $q = \beta$

In this case the following infinite-dimensional matrices satisfy the Eqs. (11)–(17):

$$\mathbf{L} = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & \dots \\ 0 & 0 & 1 & 1 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \qquad \mathbf{R} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & \dots \\ 0 & 0 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \tag{26}$$

$$\mathbf{E} = \beta/\alpha \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & s & 0 & 0 & \dots \\ 0 & 0 & s^2 & 0 & \dots \\ 0 & 0 & 0 & s^3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \tag{27}$$

Then the densities of particles and the currents for finite $N$ can be found using Eqs. (18)–(21). To find the currents in the large $N$ limit we have to calculate the asymptotic behavior of $\langle W \mid \mathbf{G}^N \mid V \rangle$. Recall that from the Eqs. (26)–(27) the matrix $\mathbf{G}$ can be chosen as

$$\mathbf{G} = \begin{pmatrix} 2+(\beta/\alpha) & 1 & 0 & 0 & \dots \\ 1 & 2+(\beta/\alpha)s & 1 & 0 & \dots \\ 0 & 1 & 2+(\beta/\alpha)s^2 & 1 & \dots \\ 0 & 0 & 1 & 2+(\beta/\alpha)s^3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \tag{28}$$

We could not calculate the asymptotic value of $\langle W \mid \mathbf{G}^N \mid V \rangle$ as $N \to \infty$ for all possible values of the parameter $s$ $(0 \leqslant s \leqslant 1)$, only for the two limiting cases $s = 0$ $(\beta = 1)$ and $s \to 1 (\beta \to 0)$. For the case $s = 0$ $(\beta = 1)$ we can use the results of EFGM [8]. One can find that in the limit $N \to \infty$, the matrix element $\langle W \mid \mathbf{G}^N \mid V \rangle$ has the following asymptotic forms:

(1) For $\alpha > 1$,

$$\langle W \mid \mathbf{G}^N \mid V \rangle \simeq \frac{4^{N+1}}{\sqrt{\pi} N^{3/2} (1 - 1/\alpha)^2}. \tag{29}$$

(2) For $\alpha = 1$,

$$\langle W \mid \mathbf{G}^N \mid V \rangle \simeq \frac{4^N}{\sqrt{\pi} N^{1/2}}. \tag{30}$$

(3) For $\alpha < 1$,

$$\langle W \mid \mathbf{G}^N \mid V \rangle \simeq (1 - \alpha^2) \left[ \frac{(\alpha+1)^2}{\alpha} \right]^N. \tag{31}$$

Using these results we calculate the asymptotic currents and identify two phases:
(a) For $\alpha \geqslant 1$,

$$J^+ = J^- = \tfrac{1}{4}. \tag{32}$$

This is the maximal-current phase.
(b) For $\alpha < 1$,

$$J^+ = J^- = \frac{\alpha}{(\alpha+1)^2}. \tag{33}$$

For the case $s \to 1$ ($\beta \to 0$) we can use the method described in Appendix B of Ref. [4]. In this case the matrix $\mathbf{G}$ looks as follows:

$$\mathbf{G} = \begin{pmatrix} 2+(\beta/\alpha) & 1 & 0 & 0 & \dots \\ 1 & 2+(\beta/\alpha) & 1 & 0 & \dots \\ 0 & 1 & 2+(\beta/\alpha) & 1 & \dots \\ 0 & 0 & 1 & 2+(\beta/\alpha) & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \tag{34}$$

A simple way to compute $\langle W \mid \mathbf{G}^N \mid V\rangle$ is to represent $\mid V\rangle$ as a linear combination of eigenvectors of $\mathbf{G}$ in order to evaluate $\mathbf{G}^N \mid V\rangle$. It can be checked that $\mathbf{G} \mid \theta\rangle = (2+\beta/\alpha+2\cos\theta) \mid \theta\rangle$ where $\mid \theta\rangle$ is given by

$$\mid \theta\rangle = \begin{pmatrix} \sin\theta \\ \sin2\theta \\ \sin3\theta \\ \vdots \end{pmatrix}. \tag{35}$$

Then the vector $\mid V\rangle$ can be written as a linear combination of these eigenvectors,

$$\mid V\rangle = \int_{-\pi}^{\pi} \frac{d\theta}{\pi} \sin\theta \mid \theta\rangle. \tag{36}$$

Using the fact that $\mid \theta\rangle$ is an eigenvector of $\mathbf{G}$ with eigenvalue $2+\beta/\alpha+2\cos\theta$, it follows from the Eq. (36) that

$$\langle W \mid \mathbf{G}^N \mid V\rangle = \int_{-\pi}^{\pi} \frac{d\theta}{\pi} \sin^2\theta\,(2+\beta/\alpha+2\cos\theta)^N. \tag{37}$$

The calculations predict the existence of only one phase with asymptotic forms

$$\langle W \mid \mathbf{G}^N \mid V\rangle \simeq (4+\beta/\alpha)^N (2\pi^2/3), \tag{38}$$

$$J = \frac{\alpha\beta}{(4\alpha+\beta)} \approx \beta/4. \tag{39}$$

Consequently, there is an upper value of the parameter $s$, i.e. some $s_c < 1$ such that the two-phase region exists only for $0 \leqslant s < s_c$.

### 2.4. Solution for $q = 1$ and $s = 0$

To solve this problem we define two matrices $\tilde{\mathbf{L}}$ and $\tilde{\mathbf{R}}$ as follows:

$$\tilde{\mathbf{L}} = \mathbf{L}, \qquad \tilde{\mathbf{R}} = \mathbf{R} + \mathbf{E}. \tag{40}$$

This is a convenient choice because

$$\mathbf{G} = \mathbf{L} + \mathbf{R} + \mathbf{E} = \tilde{\mathbf{L}} + \tilde{\mathbf{R}}. \tag{41}$$

Then the matrix Eqs. (11)–(17) can be expressed as

$$\langle W \mid \tilde{\mathbf{R}} = \left(\frac{1}{\beta} + \frac{1}{\alpha}\right) \langle W \mid, \tag{42}$$

$$\tilde{\mathbf{L}}\tilde{\mathbf{R}} = \tilde{\mathbf{L}} + \tilde{\mathbf{R}}, \tag{43}$$

$$\tilde{\mathbf{L}} \mid V\rangle = \frac{1}{\beta} \mid V\rangle. \tag{44}$$

But these equations describe the one-species dynamics of an ASEP with feeding rate $\alpha\beta/(\alpha + \beta)$ and removal rate $\beta$ [4]. A complete solution of this problem is given in Ref. [4]. Using these results we find two possible phases in the limit $N \to \infty$ :

(1) For $(\alpha + \beta) \leqslant 2\alpha\beta$. This is the maximal-current phase with the current equal to

$$J = \tfrac{1}{4}. \tag{45}$$

(2) For $(\alpha + \beta) > 2\alpha\beta$. Here the asymptotic current is given by

$$J = \frac{\alpha\beta}{(\alpha + \beta)} \left(1 - \frac{\alpha\beta}{(\alpha + \beta)}\right). \tag{46}$$

One can check that if we make another choice for the matrices $\tilde{\mathbf{L}}$ and $\tilde{\mathbf{R}}$,

$$\tilde{\mathbf{R}} = \mathbf{R}, \qquad \tilde{\mathbf{L}} = \mathbf{L} + \mathbf{E} \tag{47}$$

the results do not change.

It is interesting to note that in this case the model was studied by EFGM using mean-field analysis and Monte-Carlo simulations. They observed spontaneous symmetry breaking, i.e., the existence of phases where the currents of positive and negative particles are not equal. Our exact solution confirms the mean-field solutions, but only for symmetric phases; we do not have an asymmetric phase where the currents of the positive and negative particles would be different. The phase diagram for this set

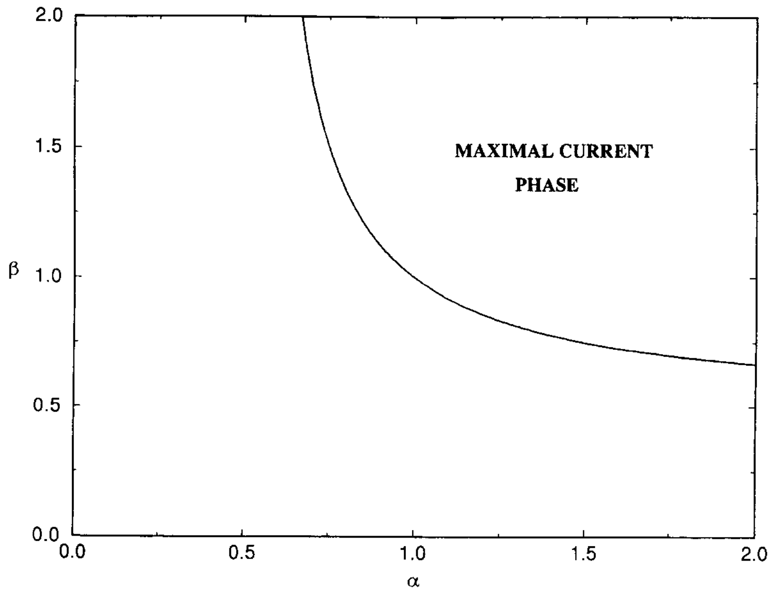


Fig. 1. Phase diagram for the case $q = 1$ and $s = 0$.

of parameters is presented in Fig. 1. It consists only of the symmetric phases of the mean-field phase diagram of EFGM [8].

## 3. Discussion

In the present paper we investigated the one-dimensional asymmetric exclusion process with two types of particles hopping with different rates in different directions. The steady-state properties of the system were studied using the matrix product method. We found exact solutions for some sets of parameters. Our solutions were also analyzed in the limit $N \to \infty$. In most cases two phases were observed, one of them a maximal-current phase in which the current is equal to $q/4$. The transitions between the phases are continuous. We compared our solutions with the results obtained by EFGM for a totally asymmetric exclusion model with two species which is a special case of our more general problem. Using the matrix product exact solution we were not able to find a spontaneous symmetry breaking as reported by EFGM. A possible explanation is that the matrix product method could give us only symmetric solutions ($J^+ = J^-$), as can be concluded from Eq. (21). It would be interesting to study the stability of the symmetric solution that we obtained in an attempt to find a spontaneous symmetry breaking.

It has recently come to my attention that some of the results and methods in Ref. [4] were obtained independently by Schütz and Domany [9].

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