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Replica-scaling analysis of diffusion in quenched correlated random media

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We propose a unifying picture of d-dimensional diffusion in randomly correlated media $\dot{Z}(\mathbf{x},t)=D\Delta Z+V(\mathbf{x})Z$ using the replica-scaling method of Zhang [Phys. Rev. B **42**, 4897 (1990)]. Here $V(\mathbf{x})$ denotes a frozen random potential of strength λ and correlation length a and arbitrary variance $\langle V(\mathbf{x})V(\mathbf{0})\rangle=\lambda^2 R_a(\mathbf{x})$. We are interested in the temporal dependence of $\langle Z^n\rangle$ and the diffusion law. It is demonstrated, in particular, that for the special case of uncorrelated disorder the asymptotic behavior is given by $\ln\langle Z^n\rangle\cong\lambda^2 a^{-d}n^2t^2$, recovering the earlier result of Zeldovich *et al.* [Sov. Phys. JETP **62**, 1188 (1985)], whereas at intermediate times $nt\leq Da^{d-2}/\lambda^2$ the situation is dimension dependent: we obtain $\ln\langle Z^n\rangle\cong(\lambda^4/D^d)^{1/(2-d)}(nt)^{(4-d)/(2-d)}$ for $d<2$ and $\ln\langle Z^n\rangle\cong -n$ for $d>4$. For $2<d<4$ we conclude that $\ln\langle Z^n\rangle\cong\lambda^2 D^{-d/2}n^2t^{2-(d/2)}$ if $t<a^2/D$. The relation between the temporal dependence of $\langle Z^n\rangle$ and the diffusion law is also established. These results are expected to be as accurate as other Flory-like theories.

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The problem of diffusion in a random environment arises in many physical and biological systems [1–4]. This task is mathematically equivalent to the models of a polymer in a random medium or of a stochastically growing interface with spatially random deposition of particles [5]. In spite of the potential importance of these systems the theoretical understanding of the properties of these objects is still incomplete.

Their dynamics is described by

$$\dot{Z}(\mathbf{x},t)=D\Delta Z+V(\mathbf{x})Z\,, \tag{1}$$

where $Z(\mathbf{x},t)$ is proportional to the density of certain physical, chemical, or biological objects, and D is the diffusion constant. We consider a random potential V that has zero mean and a nonzero and finite variance

$$\langle V\rangle=0,\ \langle V(\mathbf{x})V(\mathbf{0})\rangle=\lambda^2 R_a(|\mathbf{x}|)\,, \tag{2}$$

where $\langle\ \rangle$ denotes the average over disorder, λ^2 is the disorder strength, $R_a(|\mathbf{x}|)$ is a function of characteristic width a [for scales $x<a$ the random potential is strongly correlated and $R_a\ (x<a)=\text{const}$]; its form for $x\geq a$ depends on the nature of defects. The finiteness of a will play an important role, as will be seen below. We suppose that $R_a\ (x\geq a)=x^{\alpha}$ with arbitrary exponent α.

We are interested in the diffusion law $x=(\langle\mathbf{x}^2\rangle)^{1/2}\cong At^{\zeta}$ and moments $\langle Z^n\rangle$, for integer n, in the asymptotic long-time regime. Up to now there has been no consensus on the exponent ζ. Using a scaling argument one of us [6] concluded that

$$\zeta_{\mathrm{LR}}(\alpha)=4/(4-\alpha)\,. \tag{3}$$

This result was supported for $\alpha>\alpha_c=-d/2$ by means of a one-loop functional renormalization-group calculation [6] that used ideas of Fisher [7] and Halpin-Healy [8]. For $\alpha\leq\alpha_c$ and also for an uncorrelated potential one obtained [6]

$$\zeta_{SR}(d)=\zeta_{LR}(a_c=-d/2)=8/(8+d)\,,$$

which indicates that $d=8$ is the upper critical dimension. However, this last result contradicts previous predictions [1,5] that $\zeta=1$ for $d<4$ (with logarithmic corrections), based on Flory-type arguments [5], one-loop renormalization-group treatment of related noise-driven Burgers equation [5], and a connection to the localization of a quantum particle [1,5]. In all cases the role of higher-loop approximations is unclear. Moreover, Nattermann and Renz [5] predict for $d>4$ a phase transition from the usual diffusion law $x^2\cong Dt$ to a strong-coupling regime with unknown $\zeta>\frac{1}{2}$ as the disorder degree is changed.

The correct temporal dependence of $\langle Z^n\rangle$ is also under debate. In early work Zeldovich *et al.* [2] suggested that independent of the space dimension,

$$\ln\langle Z^n\rangle=\text{const}\times\lambda^2n^2t^2,\quad t\rightarrow\infty\,,$$

for a lattice version of original problem (1) with rapidly decreasing disorder. However, the exact solution [4] of a continuum model (1) with $R(x)=\delta(x)$ for the special case of $d=n=1$ led to the conclusion

$$\ln\langle Z\rangle=\text{const}\times\lambda^4t^3/D,\quad t\rightarrow\infty\,,$$

which evidently contradicts Zeldovich *et al.* [2]. This discrepancy was resolved by Rosenbluth [9] who demonstrated that both solutions [2,4] make sense: If the small-scale cutoff is present, the asymptotic temporal dependence of $\langle Z\rangle$ obeys the Zeldovich *et al.* [2] result; whereas if this cutoff goes to zero the exact continuum result of Tao [4] is reproduced. Rosenbluth [9] also showed that for the one-dimensional case in the continuum limit

$$\ln\langle Z^n\rangle=\text{const}\times\lambda^4n^3t^3/D,\quad t\rightarrow\infty\,,$$

and $\ln\langle Z\rangle$ is proportional to $t^{(4-d)/(2-d)}$ for $d<2$. However, his method does not work for $d>2$ and does not reproduce the properties of usual diffusion if the disorder parameter is put to zero.

The case of arbitrary space dimension was also studied by Tao, Widom, and Webman [10]. They claimed that in the case of uncorrelated disorder for $d>4$ the usual diffusion law holds, whereas for $d<4$

$$\ln\langle Z\rangle=\text{const}\times\lambda^2t^{2-(d/2)}/D^{d/2},\quad t\rightarrow\infty\,.$$

Curiously, this coincides with the initial incorrect result of Tao [3] for $d=1$ but contradicts both Rosenbluth [9] and the exact solution [4].

The aim of the present paper is to give a comprehensive description of the system for all values of D, λ, d, a, and n and to resolve all the contradictions mentioned. Our treatment is inspired by recent work of Zhang [11] on the related problem of interfaces in random correlated media. It will be shown below that almost all the quoted findings indeed take place, provided that the appropriate conditions are fulfilled. We also demonstrate that the asymptotic temporal dependence of $\langle Z^n\rangle$ enables us to reproduce the diffusion law $x\cong At^{\zeta}$. In what follows we systematically drop numerical factors of the order of unity; moreover, $\langle Z^n\rangle$ is calculated with exponential accuracy.

With the initial condition $Z(\mathbf{x},0)=\delta(\mathbf{x})$ Eq. (1) has the formal solution [1-5,9,10]

$$Z(\mathbf{x},t)=\int_{x(0)=0}^{x(t)=x}D\mathbf{x}(t)\exp\left[-\int_0^t dt[(\dot{\mathbf{x}}^2/4D)-V(\mathbf{x})]\right].\tag{4}$$

Equation (4) can be interpreted as the restricted partition function of a d-dimensional Gaussian polymer of length t in a random potential or a $(d+1)$-dimensional directed polymer in a random potential, which is correlated along the t axis. Our results are simpler to understand in terms of the polymer picture. The transformation $F(\mathbf{x},t)=-\ln Z(\mathbf{x},t)$ determines the polymer free energy. In the polymer model the diffusion law $x\cong At^{\zeta}$ connects the characteristic transverse deviation x of a $(d+1)$-dimensional directed polymer (or the mean end-to-end distance x for a d-dimensional Gaussian polymer) with its total length t. The free-energy fluctuation ΔF is also of interest. There is a standard assumption that

$$\Delta F\cong A^2t^{2\zeta-1}/D\,,\tag{5}$$

which follows from a dimensional argument about the kinetic term in the Hamiltonian corresponding to the partition function (4) ($x^2/Dt\cong A^2t^{2\zeta-1}/D$).

Let us raise the partition function (4) to the nth power and average it over disorder (2)

$$\langle Z^n\rangle=\exp\langle(-nF)\rangle=\int D\mathbf{x}\exp\left[-\sum_{i=1}^{n}\int(\dot{\mathbf{x}}_i^2/D)dt+\lambda^2\sum_{i,j}^{n}\int\int dt\,dt'R_a(|\mathbf{x}_i-\mathbf{x}_j'|)\right].\tag{6}$$

This describes n identical polymers, or replicas, with mutual interaction $R_a(|\mathbf{x}_i-\mathbf{x}_j'|)$ subject to the initial condition $\mathbf{x}_i(0)$.

Before actually tackling this path integral (6) we show, following Zhang [11], how to use the results to obtain the exponent ζ and the prefactor A. Suppose that (6) is of the form $\exp(Bn^{\beta}t^{\gamma})$, i.e.,

$$\int dF\,P(F)\exp(-nF)=\langle\exp(-nF)\rangle=\exp(Bn^{\beta}t^{\gamma})\,,\tag{7}$$

then the right-hand side of (7) can be regarded as the Laplace transform of the probability distribution density $P(F)$ of the free energy of a *single* polymer in the random medium. Inverting (7) in the limit $t\rightarrow\infty$ gives rise to the free-energy distribution density

$$P(F)\cong\exp[-(|\Delta F|^{\beta}/Bt^{\gamma})^{1/(\beta-1)}]\,,$$

where we set $\langle F\rangle=0$. Knowledge of $P(F)$ allows us to determine the free-energy fluctuation

$$\Delta F\cong B^{1/\beta}t^{\gamma/\beta}\,,$$

which can be combined with (5) to yield ζ and A:

$$\zeta=\tfrac{1}{2}+(\gamma/2\beta),\quad A^2\cong DB^{1/\beta}.\tag{8}$$

The first of these results was first obtained by Zhang [11] in a slightly different context.

Now we show how to calculate the average of the replicated partition function (6). Let us introduce, for convenience, the corresponding free energy. Minimization of F_n with respect to the average distance x between replica ends allows us to find $\langle Z^n \rangle$. From (6) we obtain the estimate

$$F_n \cong n(x^2/Dt + Dt/x^2) - n^2\lambda^2 t^2 x^\alpha, \quad x \geq a . \tag{9}$$

The first term is the elastic energy of a distorted polymer, the second is an entropic repulsion among replicas confined into a well of characteristic size x [11,12], and the third term is the defect-induced interaction between replicas. Note that in the case $\lambda^2=0$ minimization of (9) with respect to x leads to the free Gaussian polymer result (the usual diffusion law):

$$x_G^2 \cong Dt . \tag{10}$$

Therefore, the simultaneous inclusion of the two first terms in Eq. (9) is indeed necessary.

For $\alpha > 0$ the third term in (9) corresponds to a repulsion among replicas. In this situation the minimum of (9) for $t \to \infty$ is determined by the first and third terms. Then entropic repulsion is irrelevant and we obtain the saddle-point solution $x \cong (\ln \lambda^2 Dt^3)^{1/(2-\alpha)}$ that is valid in the range $0 \leq \alpha < 2$. Substitution of this into Eq. (9) enables us to find $\langle Z^n \rangle$ and to extract ζ and A from (8)

$$\begin{aligned} \ln\langle Z^n \rangle &\cong (\lambda^4 D^\alpha)^{1/(2-\alpha)} n^{(4-\alpha)/(2-\alpha)} t^{(4+\alpha)/(2-\alpha)} , \\ x &\cong (\lambda D)^{2/(4-\alpha)} t^{4/(4-\alpha)} , \end{aligned} \tag{11}$$

thus reconfirming the scaling result [6]. However, (11) is not the asymptotic limit. Indeed, for α positive ζ is greater than unity; that is, the polymer is stretched. This implies that the Hamiltonian in Eq. (4) is unstable with respect to infinitely many gradient terms. However, this change in the partition function appears only for $x \geq t$. Therefore, for times $t \leq (\lambda D)^{-2/\alpha}$ result (11) is applicable. It is quite probable that in the asymptotic temporal regime ζ equals 1. This suggestion will be supported below.

The case $\alpha < 0$ is more interesting since the potential energy in (6) is now attractive. Therefore, we conclude that the bordering value of α for which the scaling relations (3) and (11) have a meaning is quite naturally [11] $\alpha_c = 0$, in contrast to the perturbative result [6] $\alpha_c = -d/2$. In the region of negative α it is convenient to make the change $\alpha = -d$ to cover also the case of δ-correlated disorder.

If $d < 2$ Eq. (9) has a *single* minimum whose position is given by the expression

$$x = \max[(\lambda^2 nt/D)^{1/(d-2)}, a] .$$

Provided that $(\lambda^2 nt/D)^{1/(d-2)} > a$ (that is, for times $nt < Da^{d-2}/\lambda^2$), a straightforward calculation gives

$$\ln\langle Z^n \rangle \cong (\lambda^4 D^{-d})^{1/(2-d)} (nt)^{(4-d)/(2-d)} , \tag{12}$$

$$x \cong (D^{2-d}\lambda^2)^{1/(4-d)} t . \tag{13}$$

The last dependence was obtained by Nattermann and Renz [5] as the asymptotic law for $d < 4$ and for the regime of weak disorder. However, from our consideration it follows that it is valid only for $d < 2$ and for intermediate times $t < Da^{d-2}/\lambda^2$. Note that (12) and (13) match with the results for repulsive case $\alpha > 0$ (11) at $\alpha = -d = 0$. Moreover, Eq. (12) coincides with Tao [4] for $d = n = 1$ and with Rosenbluth [9] for $n = 1$ and $d < 2$ and for $d = 1$ and general n.

For $nt > Da^{d-2}/\lambda^2$ the above solution is invalid and an elementary calculation [with help of Eqs. (8) and (9)] leads to the results of Zeldovich *et al.* [2]:

$$\ln\langle Z^n \rangle \cong \lambda^2 a^{-d} n^2 t^2 , \tag{14}$$

$$x \cong (\lambda^2 D^2 a^{-d})^{1/4} t . \tag{15}$$

In this regime $\zeta = 1$ independent of space dimension and the type of disorder; therefore, for the repulsive case $-d = \alpha > 0$ we must also obtain $\zeta = 1$ at $\alpha \to 0$ due to continuity. Equation (15) was obtained by Nattermann and Renz [5] for $d < 4$ in the strong-disorder limit. However, from our consideration it follows that (15) holds for all times $t > Da^{d-2}/\lambda^2$.

For $d > 2$ Eq. (9) has *two* minima $x_1(t) = x_G(t)$ (10) and $x_2 \cong a$. Substitution of $x_1(t)$ into (9) leads to

$$F_n \cong n - \lambda^2 D^{-d/2} n^2 t^{2-d/2} . \tag{16}$$

It is evident that the situation for $d < 4$ and $d > 4$ is different.

For $2 < d < 4$, $t \to \infty$, the comparison of the last expression and result of Zeldovich *et al.* (14) shows that $x_G(t)$ is stable for times $t < a^2/D$. In this regime

$$\begin{aligned} \ln\langle Z^n \rangle &\cong \lambda^2 D^{-d/2} n^2 t^{2-d/2} , \\ x &\cong \lambda D^{(4-d)/8} t^{(8-d)/8} . \end{aligned} \tag{17}$$

Note that (17) is consistent with Tao, Widom, and Webman [10] for the special case of $n = 1$. However, the asymptotic behavior for $t > a^2/D$ obeys Zeldovich *et al.* [2] [Eqs. (14) and (15)] again.

For $d > 4$ the Gaussian polymer solution

$$\ln\langle Z^n \rangle \cong -n ,$$

$$x \cong (Dt)^{1/2}$$

is stable for times $nt < Da^{d-2}/\lambda^2$. Note that this inequality coincides with the corresponding condition for the applicability of the Tao-like result (12). For $nt \geq Da^{d-2}/\lambda^2$ we go back to the result of Zeldovich *et al.* [2] (14) and to the diffusion law (15) for $t \geq Da^{d-2}/\lambda^2$. It is necessary to stress that for $d > 4$ we differ both from Tao, Widom, and Webman [10] (who predict only the usual diffusion) and from Nattermann and Renz [5] (who predict the phase transition from the usual diffusion to the anomalous one). The impossibility of a phase transition is evident from the uniqueness of the long-time behavior of the all moments $\langle Z^n \rangle$.

In conclusion, we have constructed a unifying picture for diffusion in random correlated media using the replica-scaling analysis invented by Zhang [11]. Some remarks are necessary on the range of validity of this approach. First, it is based on the same level of assumptions as the Flory polymer theory [12], so we expect our results are as good as the Flory-like ones. Second, Zhang's manipulation involving the Laplace transform gives a purely symmetric free-energy probability distribution. There is recent evidence [13,14] that, at least for the case of directed polymers in uncorrelated random media, the free-energy probability distribution is *asymmetric*. Therefore, one can suspect that Zhang's analysis, even at this level, is approximate.

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