Model of the hydrophobic interaction

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The potential of mean force between interstitial solute molecules in Ben-Naim's one-dimensional, many-state lattice model (related to the one-dimensional q-state Potts model) is calculated. Since the model is exactly soluble, all results are explicit and analytic. It is found that the magnitude of the effective attractive force between solutes and the range of that attraction vary inversely with each other: the strength of the attraction, as expected, increases with increasing magnitude of the (entropically unfavorable) free energy of "hydrogen-bond" formation, but at the same time the range decreases. Conversely, when the unfavourable entropy and favourable energy of "hydrogen-bond" formation are nearly in balance, the attraction between hydrophobes, while then weak, is of very long range. It is remarked that solubility in a one-dimensional solution model, when the direct intermolecular interactions are of short range, can only be defined osmotically. The solubility of the hydrophobe, as so defined, is calculated with the present model. It is found to decrease with increasing temperature, as expected for a hydrophobic solute.

1   Introduction

The hydrophobic effect (thinking of water as the solvent) is a manifestation of an unfavorable free energy of solvation. The volume of solvent that is unfavorably affected by the forced accommodation of two hydrophobic solute molecules is less when the latter are close together than when they are widely separated, thus leading to an effective, solvent-mediated attraction between them. This is superimposed on, and often dominates, the direct interaction between the solute molecules. In water, the unfavorable solvation free energy arises from an unfavorable entropy change associated with the reorganization of the solvent structure (the formation or strengthening of hydrogen bonds). This unfavourable entropy change outweighs the accompanying energy change, which by itself would favor that structural rearrangement.

 Ben-Naim[1] has described a one-dimensional, many-state lattice model (related to the "q-state Potts model"[2] in one-dimension), which we now adapt to illustrate and illuminate the hydrophobic effect. The model is suited to this purpose because it incorporates the basic mechanism described above, even though its picture of "water" is highly unrealistic. There are very many interesting and important studies of the hydrophobic effect that are based on much more realistic pictures of water.[3–16] The advantages of the present model are that it is analytically soluble, and that its extreme simplicity allows some aspects of the hydrophobic effect that might otherwise be obscured by detail to be exhibited clearly.

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In the following section we define the model and outline the calculation of the solvent-mediated contribution to the potential of mean force between solute molecules. In Section 3 we calculate, *via* the model's transfer matrix, the elements called for in Section 2 and thus obtain explicit formulas for the mean force. Section 4 provides numerical illustrations of the results. In Section 5 we define and calculate the solute's solubility in the model solvent.

2 Model and the potential of mean force

The model one-dimensional solvent is shown in Fig. 1. Each molecule occupies one lattice site and may be in any of q states (orientations). The horizontal orientation is identified as state number 1. A hydrophobic solute molecule (shown as the shaded circle in the figure) can only be accommodated at an interstitial site between two solvent molecules that are both in state 1. Thus, the presence of such a solute forces its two neighboring solvent molecules to be both in state 1. A pair of neighboring solvent molecules with both members of the pair in state 1 corresponds to a "hydrogen-bond." Only neighboring solvent molecules interact with each other, and they interact also with any interstitial solute that may reside between them.

Let w_{ij} $(i, j = 1, \ldots, q)$ be the energy of interaction between a pair of neighboring solvent molecules when one is in state i and the other in state j, and take

$$w_{ij} = \begin{cases} w, & \text{when } i = j = 1 \\ u, & \text{otherwise} \end{cases} \tag{1}$$

with

$$u > w \tag{2}$$

The positive difference $u - w$ may be identified as the (magnitude of) the favorable energy of "hydrogen-bond" formation, while the positive quantity $k \ln(q - 1)$, where k is Boltzmann's constant, is the (magnitude of) the unfavorable entropy. They are the energy and entropy associated with restricting a solvent molecule to be in state 1, given that a specified one of its neighbors is also in state 1.

Let v be the energy of interaction of an interstitial solute molecule with its solvent neighbors. The model is then defined by three parameters, $u - w$, v, and q. It will transpire that the solvent-mediated part of the potential of mean force between solutes depends only on the first and third of these (thus, only on the properties of the pure solvent, as already remarked by Pratt and Chandler[5]), while the solubility of the solute depends on all three.

In this model the picture of "water" is highly oversimplified, and in this simplest version of the model there is no provision for distinguishing solutes of different sizes. Much of the detail incorporated in more realistic models[3–16] is therefore absent, although the essential features of the hydrophobic interaction are still present.

The calculation of the potential of mean force will be based on the potential-distribution theorem.[17] Let $\phi(r)$, with r measured in units of the lattice spacing, be the direct interaction potential between two solute molecules a distance r apart and let $g(r)$ be the solute–solute pair-distribution function. Also, let P_{11} be the probability that the solvent molecules at a given pair of consecutive sites be both in state 1 irrespective of the states of the molecules at the other sites, and let $P(r)$, likewise, be the probability that the molecules at two such pairs of consecutive sites m, $m + 1$ and n, $n + 1$ with $n - m = r \geqslant 1$ be all in state 1. Then in the dilute-solution limit, where
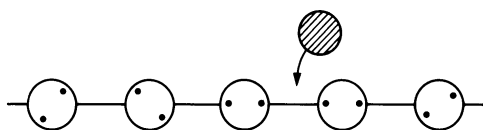

Fig. 1 The model one-dimensional solvent, with each molecule centered at a lattice site and in one of q states (orientations). The horizontal orientation is identified as state 1. A solute molecule (the shaded circle in the figure) can only be accommodated at interstitial sites between two solvent molecules that are both in state 1.

the solute "test particle" and "test-particle" pair[17] encounter only solvent but not other solutes,

$$g(r)\mathrm{e}^{\phi(r)/kT} = \frac{P(r)\mathrm{e}^{-2v/kT}}{(P_{11}\,\mathrm{e}^{-v/kT})^2} = \frac{P(r)}{P_{11}^2} \qquad (r \geqslant 1) \tag{3}$$

at the temperature T. We note the cancellation of the parameter v in this expression.

The total potential of mean force between pairs of solute molecules is $-kT \ln g(r)$, of which the solvent-mediated part, $W(r)$, is $-kT \ln g(r) - \phi(r)$; thus,

$$W(r) = -kT \ln[g(r)\mathrm{e}^{\phi(r)/kT}]$$

$$= -kT \ln[P(r)/P_{11}^2] \tag{4}$$

from eqn. (3). That this is independent of the parameter v and so depends only on the properties of the pure solvent is as anticipated. Explicit expressions for P_{11} and $P(r)$, which are the two ingredients in the formula for $W(r)$, are obtained in the following.

3   Transfer matrix and calculation of $W(r)$

The quantities P_{11} and $P(r)$ required in eqn. (4) for the evaluation of $W(r)$ may be obtained from the eigenvalues and eigenvectors of the transfer matrix V of the model solvent. Let

$$a = \mathrm{e}^{-u/kT}, \qquad b = \mathrm{e}^{-w/kT} \tag{5}$$

with u and w the interaction energies in eqn. (1). Then V is the $q \times q$ matrix in which the 1,1 element is b and all the other elements are a:

$$V = \begin{pmatrix} b & a & a & \cdots & a \\ a & a & a & \cdots & a \\ a & a & a & \cdots & a \\ \vdots & \vdots & \vdots & & \vdots \\ a & a & a & \cdots & a \end{pmatrix}_{q \times q} \tag{6}$$

It is convenient, only because it allows a slightly simpler notation in what follows, to take the linear chain of Fig. 1 to form a closed circle of N sites, so that sites 1 and N are neighbors. This has no effect on properties calculated in the thermodynamic limit $N \to \infty$ [the limit taken at fixed r in the case of $P(r)$]. Then by standard transfer-matrix methods[18] one finds that the quantity P_{11} defined in Section 2 as the probability that the molecules at a pair of consecutive sites be both in state 1, is

$$P_{11} = \frac{1}{Z} V_{11}(V^{N-1})_{11} \tag{7}$$

where Z is the partition function of the model solvent,

$$Z = \mathrm{trace}\ V^N \tag{8}$$

and where in eqn. (7) the notation V_{11} and $(V^{N-1})_{11}$ means the 1,1 element of the matrices V and V^{N-1}, respectively. Likewise, $P(r)$, the probability that the molecules at two such pairs of sites m, $m+1$ and n, $n+1$, with $n-m=r$, be all in state 1, is

$$P(r) = \frac{1}{Z}(V_{11})^2(V^{r-1})_{11}(V^{N-r-1})_{11} \qquad (r \geqslant 1) \tag{9}$$

Note the distinction between $(V_{ij})^p$ and $(V^p)_{ij}$ when $p > 1$. The relation in eqn. (9) holds, as indicated, even down to $r = 1$, where the sites $m+1$ and n are the same site.

V as defined in eqn. (6) has two positive eigenvalues,[1] $\lambda^{(1)} > \lambda^{(2)} > 0$, the remaining $q-2$ eigenvalues being all 0. Let $\psi_i^{(v)}$ be the ith component ($i = 1, \ldots, q$) of the eigenvector that belongs to the eigenvalue $\lambda^{(v)}$ ($v = 1, \ldots, q$). Then from the structure of V in eqn. (6) it follows, since $\lambda^{(1)}$ and $\lambda^{(2)}$ are non-zero, that all the $\psi_i^{(1)}$ except $\psi_1^{(1)}$ have a common value, as do all the $\psi_i^{(2)}$ except $\psi_1^{(2)}$;

i.e.,

$$\psi_2^{(1)} = \psi_3^{(1)} = \cdots = \psi_q^{(1)}; \qquad \psi_2^{(2)} = \psi_3^{(2)} = \cdots = \psi_q^{(2)} \tag{10}$$

Then the eigenvalues $\lambda^{(1)}$ and $\lambda^{(2)}$ of V and their associated eigenvectors satisfy[1]

$$\left. \begin{array}{l} b\psi_1^{(v)} + (q-1)a\psi_2^{(v)} = \lambda^{(v)}\psi_1^{(v)} \\ a\psi_1^{(v)} + (q-1)a\psi_2^{(v)} = \lambda^{(v)}\psi_2^{(v)} \end{array} \right\} (v = 1, 2) \tag{11}$$

From eqn. (11), $\lambda^{(1)}$ and $\lambda^{(2)}$ are, respectively, the larger and smaller of the two roots λ (both positive) of[1]

$$\begin{vmatrix} b - \lambda & (q-1)a \\ a & (q-1)a - \lambda \end{vmatrix} = 0 \tag{12}$$

Take the eigenvectors to be normalized by $\Sigma_{i=1}^{q} \psi_i^{(v)2} = 1$, so that, from eqn. (10),

$$\psi_1^{(v)2} + (q-1)\psi_2^{(v)2} = 1 \qquad (v = 1, 2) \tag{13}$$

Then the distinct components $\psi_1^{(1)}$, $\psi_2^{(1)}$ and $\psi_1^{(2)}$, $\psi_2^{(2)}$ of the normalized eigenvectors follow from eqns. (11)–(13).

The elements of any power V^p of V may be expressed in terms of the eigenvalues $\lambda^{(v)}$ and normalized eigenvectors $\psi^{(v)}$ ($v = 1, \ldots, q$) by

$$(V^p)_{ij} = \sum_{v=1}^{q} \lambda^{(v)p} \psi_i^{(v)} \psi_j^{(v)} \tag{14}$$

But we noted that $\lambda^{(1)} > \lambda^{(2)} > 0$ and all other $\lambda^{(v)} = 0$. It then follows from this and from eqns. (6)–(9), upon taking the thermodynamic limit $N \to \infty$ [at fixed r in the case of $P(r)$], that

$$P_{11} = \frac{b}{\lambda^{(1)}} \psi_1^{(1)2} \tag{15}$$

and

$$P(r \geqslant 1) = P_{11}^2 \left[1 + \left(\frac{\psi_1^{(2)}}{\psi_1^{(1)}} \right)^2 \left(\frac{\lambda^{(2)}}{\lambda^{(1)}} \right)^{r-1} \right] \tag{16}$$

The formula in eqn. (15) for P_{11} will be required in Section 5 in the calculation of the solubility of the solute in the model solvent. Meanwhile, from eqns. (4) and (16), we obtain for the solvent-mediated part, $W(r)$, of the potential of mean force between two solute molecules separated by r ($\geqslant 1$) lattice spacings, in the dilute-solution limit,

$$W(r) = -kT \ln \left[1 + \left(\frac{\psi_1^{(2)}}{\psi_1^{(1)}} \right)^2 \left(\frac{\lambda^{(2)}}{\lambda^{(1)}} \right)^{r-1} \right] \tag{17}$$

One sees from eqn. (17) that $W(r)$ vanishes proportionally to $\exp(-r/\xi)$ at large r, with an exponential decay length ξ, which we may identify as the range of the solvent-mediated force between solutes, given by

$$\xi = \frac{1}{\ln \dfrac{\lambda^{(1)}}{\lambda^{(2)}}} \tag{18}$$

This is the same formula as that for the correlation length in the Ising model when $\lambda^{(1)}$ and $\lambda^{(2)}$ are the largest and next largest eigenvalues of its transfer matrix.[18] Approach to the critical point in the Ising model is characterized by a closing of the gap between $\lambda^{(1)}$ and $\lambda^{(2)}$ and a consequent divergence of ξ. In the present model, it will transpire, there is an analogous phenomenon: when the favorable energy and unfavorable entropy of "hydrogen-bond" formation are nearly in balance, $\lambda^{(1)}$ and $\lambda^{(2)}$ are nearly equal and the potential of mean force is then long ranged (although weak).

The explicit calculation of $W(r)$ from eqn. (17) requires the ratios $(\psi_1^{(2)}/\psi_1^{(1)})^2$ and $\lambda^{(2)}/\lambda^{(1)}$. Define new quantities c, x, S, and Q in terms of a, b, and $q-1$ by

$$c = \frac{b}{a} = e^{(u-w)/kT} > 1 \tag{19}$$

$$x = \frac{q-1}{c} \tag{20}$$

$$S = \sqrt{1 - \frac{4x}{(1+x)^2}\left(1 - \frac{1}{c}\right)} \tag{21}$$

$$Q = \frac{\text{sign }(x-1)}{\sqrt{1 + \dfrac{4x}{(x-1)^2 c}}} \tag{22}$$

where sign $(x-1)$ means $+1$ if $x > 1$ and -1 if $x < 1$. That $c > 1$ is a consequence of the earlier assumption in (2). Then S and Q are manifestly real, with $0 < S < 1$ and $-1 < Q < 1$. Then from eqns. (11)–(13) we find for the required ratios,

$$\left(\frac{\psi_1^{(2)}}{\psi_1^{(1)}}\right)^2 = \frac{1+Q}{1-Q}, \qquad \frac{\lambda^{(2)}}{\lambda^{(1)}} = \frac{1-S}{1+S} \tag{23}$$

Then from eqns. (17) and (23), with the definitions in eqns. (19)–(22), we obtain $W(r)$ as an explicit function of r and of the model's two parameters $u-w$ and q, at any temperature T:

$$W(r) = -kT \ln\left[1 + \frac{1+Q}{1-Q}\left(\frac{1-S}{1+S}\right)^{r-1}\right] \tag{24}$$

It was remarked in Section 2 that $u-w$ may be identified as the (magnitude of) the favorable energy of "hydrogen-bond" formation and $k \ln(q-1)$ the (magnitude of) the unfavorable entropy. Then the corresponding free energy of "hydrogen-bond" formation, ΔF (unfavorable when it is positive), is $-(u-w) + kT \ln(q-1)$; or, from the definition of x in eqn. (20),

$$\Delta F = kT \ln x \tag{25}$$

We may then anticipate that for this to be a model of the hydrophobic interaction we shall need x to be greater than 1. In the numerical illustration in Section 4, which follows, it will be seen that realistic values of x are in the range 2–20.

At the same time it will be seen in Section 4 that the favorable energy and unfavorable entropy are not far from being in balance; i.e., that the positive ΔF is typically much smaller than either of the positive quantities $u-w$ or $kT \ln(q-1)$ separately; and that, as a consequence, c as defined in eqn. (19) is typically thousands of times as great as x. Thus, as a practical matter, from the definitions of S and Q in eqns. (21) and (22) the quantities $(1+Q)/(1-Q)$ and $(1-S)/(1+S)$ required in eqn. (24) may be taken to be

$$\frac{1+Q}{1-Q} \approx \begin{cases} \dfrac{(x-1)^2}{x}\,c, & x > 1 \\ 0, & x < 1 \end{cases} \tag{26}$$

$$\frac{1-S}{1+S} \approx \begin{cases} \dfrac{1}{x}, & x > 1 \\ x, & x < 1 \end{cases} \tag{27}$$

It was remarked earlier that $0 < S < 1$ (whatever x and c), so we have $0 < (1-S)/(1+S) < 1$. Therefore, from eqns. (24) and (26), the solvent-mediated attraction between solutes is practically nil when $x < 1$. This confirms that $x > 1$ [or $\Delta F > 0$, from eqn. (25)] is necessary for an attractive hydrophobic interaction, as anticipated.

When x does exceed 1, so that there is a non-negligible solvent-mediated hydrophobic attraction, we see from eqns. (18), (23), (25), and (27) that its range is

$$\xi \approx \frac{1}{\ln x} = \frac{kT}{\Delta F} \tag{28}$$

The closer the otherwise positive ΔF is to 0 the weaker is the attraction, but then also the longer ranged is it, according to eqn. (28). Thus, the strength and range of the attraction are inversely related. This will be very clear in the plots displayed in Section 4. Should this prove to be a general feature of the hydrophobic effect, not limited to the present model, it could have important implications in the interpretation of the hydrophobic effect in real systems.

When the energy and entropy of hydrogen-bond formation are nearly in balance, ΔF is near 0 and x is near 1. That, from eqns. (23) and (27), makes the eigenvalues $\lambda^{(1)}$ and $\lambda^{(2)}$ nearly degenerate. This is the analogy with approach to a critical point that was mentioned earlier. Strictly, however, as noted above, $(1 - S)/(1 + S) < 1$, so, unlike at a true critical point, we cannot here have an exact degeneracy.

4 Numerical illustration

For a realistic representation of hydrogen-bond energy one should think of the model parameter $u - w$ as being around 25 kJ mol^{-1}. For purposes of illustrating the workings of the present model we shall then take $(u - w)/k = 3000$ K. There is no generally agreed on value of the free energy of hydrogen-bond formation, except that at 300 K it is much less than 25 kJ mol^{-1}; *i.e.*, the favorable energy and unfavorable entropy are largely mutually compensating at that temperature. For purposes of illustration we shall take ΔF to be about 4 kJ mol^{-1} at 300 K, which, by eqn. (25), is equivalent to taking x to be around 5 at that temperature. Then from eqns. (19) and (20), with $(u - w)/k = 3000$ K, we conclude that $q - 1$ is around 110 000 and that x varies from 2 to 20 as the temperature varies from 275 to 348 K. Over this temperature range, then, the quantity c defined in eqn. (19) is enormously greater than x, as anticipated in Section 3.

That q should be many thousands has as a precedent what Andersen and Wheeler find in an analogous lattice model of closed-loop coexistence curves.[19] They have the parameters $\omega \approx 5000$ and $\omega^* \approx (\pi\omega)^{1/2} \sin \theta$ with θ the tetrahedral angle, $\arccos(-1/3)$, and they identify the number of orientations of each "H$_2$O" molecule as $\frac{1}{2}\omega\omega^*$. In the notation of the present model that would imply $q \approx 300 000$. Both in that model and in the present one it is probably because the picture of "water" is unrealistic that so large a value of q is required.[20]

Fig. 2 shows $W(r)/k$ as a function of r, as calculated from eqn. (24) with eqns. (19)–(22), at the three temperatures $T = 275$ K, 300 K and 348 K, with the parameter values $(u - w)/k = 3000$ K and $q - 1 = 110 000$. Using the approximations in eqns. (26) and (27) instead of the exact eqns.

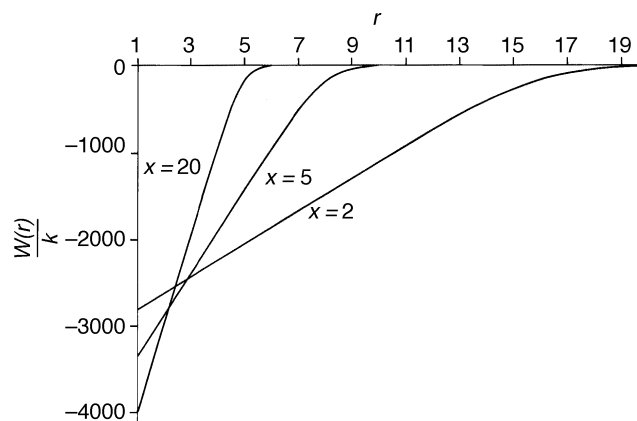**Fig. 2** $W(r)/k$ (in K), with $(u - w)/k = 3000$ K and $q - 1 = 110 000$. The curves marked $x = 2$, 5, and 20 correspond to the temperatures $T = 275$ K, 300 K, and 348 K, respectively.
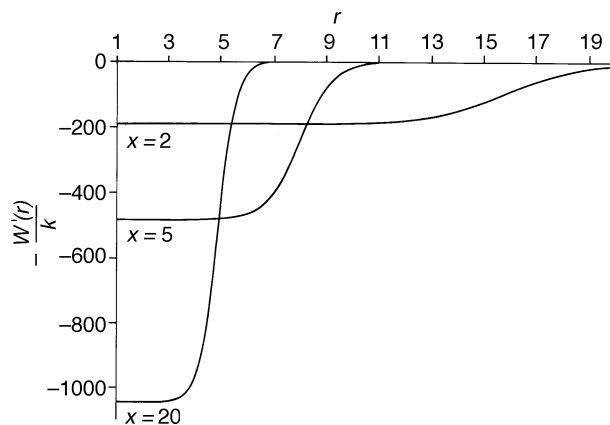
Fig. 3 $-(1/k)\,\mathrm{d}W(r)/\mathrm{d}r$ (in K), with the same values of the parameters and at the same temperatures as in Fig. 2.

(21) and (22) makes no discernible difference. The analytic function of r in eqn. (24) was used to plot smooth curves, for ease of visualization, although for this lattice model only integer values of r (as integer multiples of the lattice spacing) are defined. The plots are made only for $r \geqslant 1$; we choose not to define $W(r)$ at $r = 0$, for any such definition would be arbitrary and unphysical.

The solvent-mediated force (in units of the Boltzmann constant and of the reciprocal of the lattice constant), $-(1/k)\,\mathrm{d}W(r)/\mathrm{d}r$, is plotted in Fig. 3, with the same values of the parameters and for the same three temperatures as in Fig. 2. One sees in both Figs. 2 and 3 that the strength of the solvent-mediated hydrophobic attraction increases with increasing temperature while the range of the attraction decreases. The strength increases because the free energy ΔF, which is $-(u - w) + kT\ln(q - 1)$, becomes increasingly positive, hence increasingly unfavorable to "hydrogen-bond" formation, as T increases; while the range decreases with increasing T because then, by eqn. (28),

$$\xi \approx [-(u - w)/kT + \ln(q - 1)]^{-1} \tag{29}$$

The inverse relation between strength and range anticipated in Section 3 is apparent in the figures.

5 Solubility

Defining or measuring solubility requires the coexistence of two phases, one a reference phase and the other the solution in which the solubility of the solute is to be determined. The two phases are in equilibrium; the solute is present in the two at equal chemical potential (activity). The reference phase may be pure liquid or solid solute; it may be pure gaseous solute, in which case one is defining the solubility of a gas; or it may be another liquid or solid solution, in which case one is defining the solubility by the partition coefficient of the solute between the two phases. The case in which the reference phase is pure gaseous solute is a special case of the latter, where the "solvent" in the reference phase is vacuum.

Because normal phase coexistence is not possible in one-dimensional systems with short-range forces, the solubility of the hydrophobic solute in the model solvent can here only be defined osmotically. We imagine a semi-permeable membrane, permeable only to the solute. (Note that this is opposite to the usual picture of osmosis, where the membrane is permeable only to solvent, not to solute.) In the present one-dimensional model the "membrane" is a point, separating the saturated solution of interest on one side from the reference phase on the other.

For definiteness, and for purposes of illustration, we shall take the reference phase to be pure gaseous solute, dilute enough to be an ideal gas. We are thus determining the solubility of a gaseous hydrophobe in the model solvent. Let ρ_{gas} and ρ_{soln} be the number densities of the solute in the reference phase and in the solution, respectively, at osmotic equilibrium. We than take as
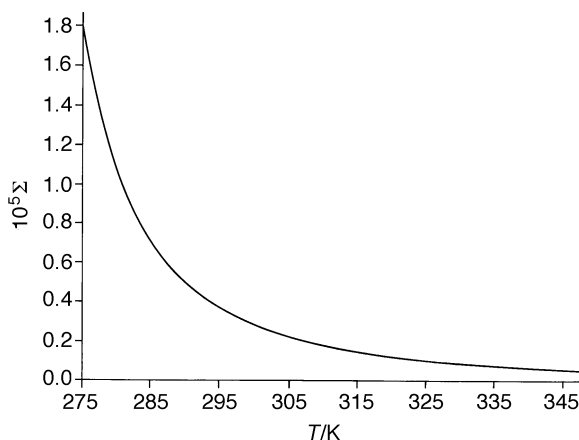
Fig. 4 Solubility, $\Sigma \times 10^5$, as a function of temperature (in K), with $(u - w)/k = 3000$ K, $q - 1 = 110\,000$, and $v = 0$.

our definition of the solubility, Σ, the dimensionless ratio

$$\Sigma = \rho_{\text{soln}}/\rho_{\text{gas}} \tag{30}$$

With the reference gas phase taken to be ideal, and with the solubility in the solvent assumed to be low enough so that the saturated solution is also very dilute, this Σ is the Ostwald absorption coefficient. It is also proportional to the Henry's-law coefficient (or its reciprocal, depending on how one writes Henry's law).

With the activity z of a species defined so as to become asymptotically equal to the number density in the limit of an infinitely dilute gas, the denominator ρ_{gas} in the definition of Σ in eqn. (30) is also the common value of the activity z of the solute in the two phases. Then Σ is the ratio of the number density to the activity in the model solution. By the potential-distribution theorem,[17] that ratio in the present model is precisely the $P_{11} \exp(-v/kT)$ that occurs squared in the denominator of the first equality for $g(r) \exp[\phi(r)/kT]$ in eqn. (3); thus, from eqn. (15),

$$\Sigma = \frac{b}{\lambda^{(1)}} \, \psi_1^{(1)2} \, e^{-v/kT} \tag{31}$$

Unlike the potential of mean force, the solubility Σ depends on the model's third parameter, v, which is the energy of interaction of an accommodated solute molecule with its two neighboring solvent molecules. It is equivalently the energy of transfer of a solute molecule from the reference gas phase to an already available interstitial site in the model solvent. The energy of transfer without the stipulation "already available," when $q - 1$ is very large so that very few sites are indeed available, is $v - (u - w)$.

We may obtain the coefficient $b\psi_1^{(1)2}/\lambda^{(1)}$ of the exponential in eqn. (31) from eqns. (11)–(13) with $v = 1$, with the result that

$$\Sigma = \frac{(1/c)e^{-v/kT}}{-(x - 1)(1 - 1/c) + \frac{1}{2}(x - 1 + 2/c)[x + 1 + (x - 1)/Q]} \tag{32}$$

with c, x, and Q as given by eqns. (19), (20), and (22). With $x > 1$ and c thousands of times as great, just as before, we have $Q \approx 1$ from eqn. (22) and

$$\Sigma \approx \frac{e^{-v/kT}}{(x - 1)^2 c} \tag{33}$$

The pre-factor $[(x - 1)^2 c]^{-1}$ of the exponential in eqn. (33) is the approximation to P_{11}, the probability that the solvent molecules at a pair of consecutive sites be both in state 1. With $u - w > 0$, this probability must decrease with increasing temperature, and that this is so for

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$[(x-1)^2 c]^{-1}$ may be verified from the definitions of c and x in eqns. (19) and (20). The additional energy parameter v may be positive or negative. In Fig. 4 there is plotted the solubility Σ as a function of temperature, from eqn. (33), for $v = 0$ and for the same values of the other parameters, $q - 1$ and $u - w$, as in Figs. 2 and 3. The temperature range in Fig. 4 is 275 to 348 K; these are the lowest ($x = 2$) and highest ($x = 20$) of the temperatures to which the three curves in Figs. 2 and 3 correspond. As expected, Σ falls with increasing T. This is one of the signatures of the hydrophobic effect in real systems. The solubility is seen to be very low with these values of the parameters: of the order of 10^{-6} or 10^{-5} over the temperature range of the figure.

For $v < 0$ the solubility is greater than that in Fig. 4 by the factor $\exp(-v/kT)$, but it then decreases more rapidly with increasing T. When $v > 0$ the solubility is less by that exponential factor, but it then decreases less rapidly with increasing T, and may even go through a minimum and then increase. This happens when $v > u - w$; *i.e.*, when the energy of transfer of a solute molecule from the reference gas phase into the solvent (with $q - 1$ very large) is positive. Such a minimum in the solubility of a hydrophobic solute in water as a function of temperature is often observed in experiment and in simulation;[6] but in the present model the solubility Σ at such a minimum could only be at most of order $1/(q-1)^2$, which is unrealistically low. Thus, it may be that the mechanism by which the minimum occurs here is different from that by which it occurs in real systems. This question remains open.

Acknowledgement

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Paper 8/09308C